Theoretical Computer Science ••• (••••) •••–•••

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Theoretical Computer Science

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Structure fault tolerance of k-ary n-cube networks

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ARTICLE INFO

Article history:
Received 11 November 2018
Received in revised form 19 January 2019
Accepted 17 June 2019
Available online xxxx
Communicated by S.-y. Hsieh

Keywords:
Structure connectivity
Substructure connectivity
Fault tolerance
k-ary n-cube

ABSTRACT

Let G be a graph and H be a certain connected subgraph of G. The H-structure connectivity $\kappa(G; H)$ (resp. H-substructure connectivity $\kappa^s(G; H)$) of G is the minimum number of a set of subgraphs $F = \{H_1, H_2, \cdots, H_m\}$ (resp. $F = \{H'_1, H'_2, \cdots, H'_m\}$) such that H_i is isomorphic to H (resp. H'_i is a connected subgraph of H) for every $1 \le i \le m$, and F's removal will disconnect G. For the k-ary n-cube Q_n^k, the $\kappa(Q_n^k; H)$ and $\kappa^s(Q_n^k; H)$ were determined for $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}\}$. In this paper, we show $\kappa(Q_n^k; H)$ and $\kappa^s(Q_n^k; H)$ for $H \in \{P_l, C_l\}$ where $3 \le l \le 2n$.

1. Introduction

The design of interconnection networks is an integral part of parallel processing and distributed system. The choice of the interconnection network topology determines the performance of the system significantly. An interconnection network can be modeled as a simple connected graph with processors and links between processors as vertices and edges, respectively.

In a network, traditional connectivity is an important measure since it can correctly reflect the fault tolerance of network systems with few processors. However, it always underestimates the resilience of large networks. There is a discrepancy because the occurrence of events which would disrupt a large network after a few processor or link failures is highly unlikely. To overcome this shortcoming, several new concepts on the connectivity of graphs were posed. Harary [9] introduced the concept of conditional connectivity by imposing some conditions on the connected components. Furthermore, Latifi et al. [11] generalized the concept conditional connectivity by introducing restricted h-connectivity. Following this trend, several kinds of conditional connectivity were proposed and studied in [7,10,20,22], such as g-extraconnectivity and R_g-connectivity. The g-extraconnectivity of a graph G, denoted by $\kappa_g(G)$, is the minimum cardinality of a set of nodes in G whose deletion disconnects G and leaves each remaining component with more than g nodes. The R_g-connectivity of a graph G, denoted by $\kappa^g(G)$, is the minimum cardinality of a set of nodes in G, whose deletion disconnects G and each node of the remaining components has at least g neighbors. Later, Zhao and Yang [23,24] investigated the r-component connectivity $c\kappa_r(G)$ of a non-complete graph G, which is the minimum number of vertices whose deletion results in a graph with at least r components. By the definitions above, we know that the g-extraconnectivity, the R_g-connectivity and the r-component connectivity can be all regarded as a generalization of the traditional connectivity $\kappa(G)$.

So far, most works on reliability and fault-tolerance have focused on the effect of individual nodes becoming faulty. However in reality, nodes that are linked could affect each other, and the neighbors of a faulty node might be more vul-

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https://doi.org/10.1016/j.tcs.2019.06.013

nerable and have a higher probability of becoming faulty later. Also to be noted is that increasingly in today's technology, networks and subnetworks are made into chips. That means if any node/nodes on the chip become faulty, the whole chip can be considered faulty. All these motivate the study of fault-tolerance from the perspective of some structures instead of basing on individual nodes. Under this consideration, Lin et al. [12] introduced the concept of structure connectivity and substructure connectivity of graphs.

The k-ary n-cube, denoted by Q_n^k, is one of the most popular network topologies for building a multiprocessor system due to its desirable properties: it is node-symmetric and edge-symmetric [1]; it is Hamiltonian [2,5]. In recent years, k-ary n-cubes have been well studied and their basic topological and algorithmic properties are well understood, for example, cycle or path embedding [1,13,16,18,19], conditional diagnosability, fault tolerance [8,17,21], and so on. In this paper, we study the structure connectivity and substructure connectivity of Q_n^k. Lin et al. [12] studied $\kappa(Q_n; H)$ and $\kappa^s(Q_n; H)$ for the hypercube Q_n and $H \in \{K_{1,1}, K_{1,2}, K_{1,3}, C_4\}$. And the result was also extended to the k-ary n-cube[14]. Moreover, Sabir and Meng[15] investigated the structure fault tolerance of hypercubes and folded hypercubes, that is, $\kappa(Q_n; H)$ and $\kappa(FQ_n; H)$ for $H \in \{P_k, C_{2k}\}$. In this paper, we consider similar problems for the k-ary n-cube Q_n^k.

2. Preliminaries

For the definition and notation of graph theory, we follow [3]. The neighborhood $N_G(v)$ of a node v in a graph $G = (V, E)$ is the set of nodes adjacent to v. For $S \subset V$, the neighborhood $N_G(S)$ of S in G is defined as $N_G(S) = (\cup_{v\in S} N_G(v)) - S$. Let $P_k = \langle v_1, v_2, \cdots, v_k\rangle$ and $C_k = \langle v_1, v_2, \cdots, v_k, v_1\rangle$ be a path and a cycle of order k, respectively. And we use $P_k^{-1} = \langle v_k, v_{k-1}, \cdots, v_1\rangle$ to denote the reverse of P_k. For a subgraph H of a graph G, we use $G - H$ to denote the subgraph of G induced by $V(G) - V(H)$. For a set $F = \{H_1, H_2, \cdots, H_m\}$, where each H_i is isomorphic to a connected subgraph of G, we use $G - F$ to denote the subgraph of G induced by $V(G) - V(H_1) - V(H_2) - \cdots - V(H_m)$.

A set F of subgraphs of G is a subgraph-cut of G if $G - F$ is a disconnected or trivial graph. Let H be a subgraph of G. Then F is an H-structure-cut if F is a subgraph-cut, and every subgraph in F is isomorphic to H. The H-structure-connectivity of G, denoted by $\kappa(G; H)$, is the minimum cardinality of all H-structure-cuts of G. Then F is an H-substructure-cut if F is a subgraph-cut, such that every subgraph in F is isomorphic to a connected subgraph of H. The H-substructure-connectivity of G, denoted by $\kappa^s(G; H)$, is the minimum cardinality of all H-substructure-cuts of G.

For $k \geq 3$ and $n \geq 1$, the k-ary n-cube Q_n^k has k^n nodes, each of which has the form $x = x_{n-1}x_{n-2}\cdots x_0$ where $x_i \in \{0, 1, \cdots, k-1\}$ for $0 \leq i \leq n-1$. Two nodes $x = x_{n-1}x_{n-2}\cdots x_0$ and $y = y_{n-1}y_{n-2}\cdots y_0$ in Q_n^k are adjacent if and only if there exists an integer i such that (1) either $y_i = (x_i + 1) \bmod k$ or $y_i = (x_i - 1) \bmod k$, and (2) $x_j = y_j$ for each $j \neq i$. In this case, we say that (x, y) is an i-dimensional edge. For any node $x = x_{n-1}x_{n-2}\cdots x_0$ in Q_n^k, we set $(x)^{i+} = x_{n-1}^{i+}x_{n-2}^{i+}\cdots x_0^{i+}$ being the neighbor of x in dimension i of Q_n^k where $x_j^{i+} = x_j$ for each $j \neq i$ and $x_i^{i+} = (x_i + 1) \bmod k$, and we set $(x)^{i-} = x_{n-1}^{i-}x_{n-2}^{i-}\cdots x_0^{i-}$ being the neighbor of x in dimension i of Q_n^k, where $x_j^{i-} = x_j$ for every $j \neq i$ and $x_i^{i-} = (x_i - 1) \bmod k$. For clarity of presentation, the remaining expressions in this paper omit writing "mod k". Note that each node has degree $2n$ when $k \geq 3$ and $n \geq 1$, and Q_1^k is isomorphic to a cycle of length k.

3. $\kappa(Q_n^k; P_l)$ and $\kappa^s(Q_n^k; P_l)$

As the k-ary n-cube Q_n^k is a special class of n-demensional torus networks, we have the following two lemmas from [4].

Lemma 3.1 *([4]). If $k \neq 3$, then Q_n^k is triangle free.*

Lemma 3.2 *([4]). The k-ary n-cube Q_n^k does not contain the complete bipartite graph $K_{2,3}$.*

Lemma 3.3. *Let P_l be a path in $Q_n^k (k \geq 4)$ with $1 \leq l \leq 2n$. If v is a node of $Q_n^k - P_l$, then $|N_{Q_n^k}(v) \cap V(P_l)| \leq \lceil l/2\rceil$.*

Proof. Q_n^k is triangle free by Lemma 3.1, v can be adjacent to at most one node of any two consecutive nodes on P_l. Thus, the lemma follows. □

Lemma 3.4. *Any two nodes in Q_n^k have at most two common neighbours if they have any.*

Proof. Let u and v be two nodes in Q_n^k. Suppose that they have at least three common neighbours, then this gives a $K_{2,3}$ in Q_n^k, which is a contradiction by Lemma 3.2. Thus, the result holds. □

Lemma 3.5. *Let P_l be a path in $Q_n^k (k \geq 4)$ with $3 \leq l \leq 2n$. If u and v are two adjacent nodes of $Q_n^k - P_l$, then $|N_{Q_n^k}(\{u, v\}) \cap V(P_l)| \leq l - 1$.*

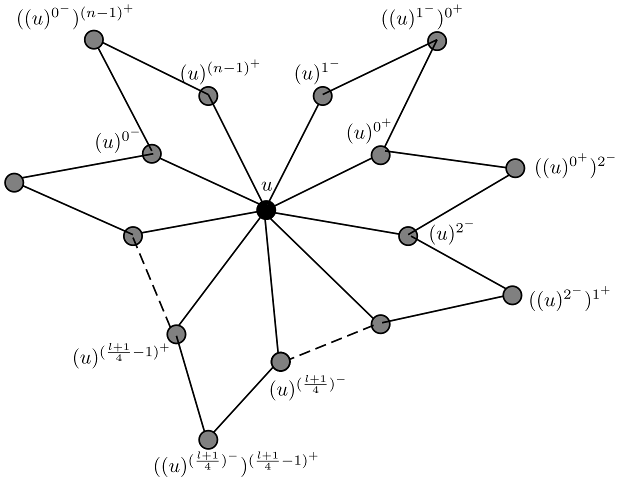

Fig. 1. A P_l-structure-cut of Q_n^k.

Proof. Clearly, $|N_{Q_n^k}(\{u, v\}) \cap V(P_l)| \leq l$. So it suffices to show that there exists at least one node on P_l, which is adjacent to neither u nor v. In deed, since Q_n^k is triangle free and by Lemma 3.4, at least one node of any three consecutive nodes on P_l is adjacent to neither u or v. It implies that $|N_{Q_n^k}(\{u, v\}) \cap V(P_l)| \leq l - 1$. □

Lemma 3.6 *([6,7]). $\kappa(Q_n^k) = 2n$ for $k \geq 3$ and $n \geq 1$; $\kappa_1(Q_n^k) = 4n - 2$ for $k \geq 4$ and $n \geq 2$.*

Lemma 3.7 *([10]). If $k \geq 4$ and $n \geq 5$, then $\kappa_2(Q_n^k) = 6n - 5$.*

Since Q_n^k is node-symmetric, we set u being an arbitrary node in Q_n^k. Note that there are $2n$ neighbors of u. A pair of neighbors u^{i+} (or u^{i-}) and u^{j+} (or u^{j-}) have a common neighbor u^{i+j+} (or u^{i+j-}, u^{i-j+}, u^{i-j-}) other than u if $i \neq j$. Moreover, the common neighbors are all distinct. So, for any set X of $\lceil \frac{l}{2} \rceil$ neighbors of u, there exists a path(or cycle) of order l that passes through all the neighbors in X. Thus, we would construct a P_l-(or C_l-)structure-cut of Q_n^k based on the idea in Lemma 3.8 and Lemma 4.1.

Lemma 3.8. *Let $3 \leq l \leq 2n$. Then $\kappa(Q_n^k; P_l) \leq \lceil \frac{4n}{l+1} \rceil$ and $\kappa^s(Q_n^k; P_l) \leq \lceil \frac{4n}{l+1} \rceil$ if l is odd; $\kappa(Q_n^k; P_l) \leq \lceil \frac{4n}{l} \rceil$ and $\kappa^s(Q_n^k; P_l) \leq \lceil \frac{4n}{l} \rceil$ if l is even.*

Proof. In the following we distinguish cases pertaining to the parity of l.

Case 1. l is odd. Without loss of generality, we assume that $l \equiv 3(mod4)$. Let $2n = q \cdot (\frac{l+1}{2}) + r$ for some nonnegative integers q and r with $0 \leq r \leq \frac{l-1}{2}$ (note that r is even). Since $2n \geq l$, we have $q \geq 1$. Let u be an arbitrary node in Q_n^k.

Case 1.1. $r = 0$. We set

$P^1 = \langle (u)^{1^-}, ((u)^{1^-})^{0^+}, (u)^{0^+}, ((u)^{0^+})^{2^-}, (u)^{2^-}, \cdots, (u)^{(\frac{l+1}{4}-1)^-}, ((u)^{(\frac{l+1}{4}-1)^-})^{(\frac{l+1}{4}-2)^+}, (u)^{(\frac{l+1}{4}-2)^+}, ((u)^{(\frac{l+1}{4}-2)^+})^{(\frac{l+1}{4})^-}, (u)^{(\frac{l+1}{4})^-}, ((u)^{(\frac{l+1}{4})^-})^{(\frac{l+1}{4}-1)^+}, (u)^{(\frac{l+1}{4}-1)^+} \rangle$,

$P^2 = \langle (u)^{(\frac{l+1}{4}+1)^-}, ((u)^{(\frac{l+1}{4}+1)^-})^{(\frac{l+1}{4})^+}, (u)^{(\frac{l+1}{4})^+}, \cdots, (u)^{(\frac{l+1}{2})^-}, ((u)^{(\frac{l+1}{2})^-})^{(\frac{l+1}{2}-1)^+}, (u)^{(\frac{l+1}{2}-1)^+} \rangle$,

$\cdots$

$P^q = \langle (u)^{(n-\frac{l+1}{4}+1)^-}, ((u)^{(n-\frac{l+1}{4}+1)^-})^{(n-\frac{l+1}{4})^+}, (u)^{(n-\frac{l+1}{4})^+}, \cdots, (u)^{0^-}, ((u)^{0^-})^{(n-1)^+}, (u)^{(n-1)^+} \rangle$.

Clearly, $P^i (1 \leq i \leq q)$ is a path of order $l = 4(\frac{l+1}{4} - 1) + 3$, which contains $\frac{l+1}{2}$ neighbors of u. Thus $F = \{P^1, P^2, \cdots, P^q\}$ forms a P_l-structure-cut of Q_n^k and $|F| = \frac{4n}{l+1}$. (See Fig. 1.)

Case 1.2. $2 \leq r \leq \frac{l-3}{2}$. We set

$P^1 = \langle (u)^{1^-}, ((u)^{1^-})^{0^+}, (u)^{0^+}, ((u)^{0^+})^{2^-}, (u)^{2^-}, \cdots, (u)^{(\frac{l+1}{4}-1)^-}, ((u)^{(\frac{l+1}{4}-1)^-})^{(\frac{l+1}{4}-2)^+}, (u)^{(\frac{l+1}{4}-2)^+}, ((u)^{(\frac{l+1}{4}-2)^+})^{(\frac{l+1}{4})^-}, (u)^{(\frac{l+1}{4})^-}, ((u)^{(\frac{l+1}{4})^-})^{(\frac{l+1}{4}-1)^+}, (u)^{(\frac{l+1}{4}-1)^+} \rangle$,

$P^2 = \langle (u)^{(\frac{l+1}{4}+1)^-}, ((u)^{(\frac{l+1}{4}+1)^-})^{(\frac{l+1}{4})^+}, (u)^{(\frac{l+1}{4})^+}, \cdots, (u)^{(\frac{l+1}{2})^-}, ((u)^{(\frac{l+1}{2})^-})^{(\frac{l+1}{2}-1)^+}, (u)^{(\frac{l+1}{2}-1)^+} \rangle$,

$\cdots$

$P^q = \langle (u)^{(n-\frac{r}{2}-\frac{l+1}{4}+1)^-}, ((u)^{(n-\frac{r}{2}-\frac{l+1}{4}+1)^-})^{(n-\frac{r}{2}-\frac{l+1}{4})^+}, (u)^{(n-\frac{r}{2}-\frac{l+1}{4})^+}, \cdots, (u)^{(n-\frac{r}{2})^-}, ((u)^{(n-\frac{r}{2})^-})^{(n-\frac{r}{2}-1)^+}, (u)^{(n-\frac{r}{2}-1)^+} \rangle$,

$P^{(q+1)} = \langle (u)^{(n-\frac{r}{2}+1)^-}, ((u)^{(n-\frac{r}{2}+1)^-})^{(n-\frac{r}{2})^+}, (u)^{(n-\frac{r}{2})^+}, \cdots, (u)^{0^-}, ((u)^{0^-})^{(n-1)^+}, (u)^{(n-1)^+}, ((u)^{(n-1)^+})^{1^-}, (((u)^{(n-1)^+})^{1^-})^{2^+}, \cdots, ((((u)^{(n-1)^+})^{1^-})^{2^+\cdots})^{(l+1-2r(modn))^+} \rangle$.

Then $F = \{P^1, P^2, \cdots, P^q, P^{(q+1)}\}$ forms a P_l-structure-cut of Q_n^k and $|F| = \lceil \frac{4n}{l+1} \rceil$. When $l \equiv 1(mod4)$, one can similarly obtain the result.

Thus, $\kappa(Q_n^k; P_l) \leq \lceil \frac{4n}{l+1} \rceil$ and consequently $\kappa^s(Q_n^k; P_l) \leq \lceil \frac{4n}{l+1} \rceil$ when l is odd.

Case 2. l is even. Without loss of generality, we assume that $l \equiv 0(mod4)$. Let $2n = q \cdot (\frac{l}{2}) + r$ for some nonnegative integers q and r with $0 \leq r \leq \frac{l-2}{2}$($r$ is even). Since $2n \geq l$, we have $q \geq 1$. We set u being an arbitrary node in Q_n^k.

Case 1.1. $r = 0$. We set

$P^1 = \langle (u)^{1^-}, ((u)^{1^-})^{0^+}, (u)^{0^+}, ((u)^{0^+})^{2^-}, \cdots, (u)^{(\frac{l}{4})^-}, ((u)^{(\frac{l}{4})^-})^{(\frac{l}{4}-1)^+}, (u)^{(\frac{l}{4}-1)^+}, ((u)^{(\frac{l}{4}-1)^+})^{(\frac{l}{4}+1)^-} \rangle$,

$P^2 = \langle (u)^{(\frac{l}{4}+1)^-}, ((u)^{(\frac{l}{4}+1)^-})^{(\frac{l}{4})^+}, (u)^{(\frac{l}{4})^+}, \cdots, (u)^{(\frac{l}{2})^-}, ((u)^{(\frac{l}{2})^-})^{(\frac{l}{2}-1)^+}, (u)^{(\frac{l}{2}-1)^+}, ((u)^{(\frac{l}{2}-1)^+})^{(\frac{l}{2}+1)^-} \rangle$,

...

$P^q = \langle (u)^{(n-\frac{l}{4}+1)^-}, ((u)^{(n-\frac{l}{4}+1)^-})^{(n-\frac{l}{4})^+}, (u)^{(n-\frac{l}{4})^+}, \cdots, (u)^{0^-}, ((u)^{0^-})^{(n-1)^+}, (u)^{(n-1)^+}, ((u)^{(n-1)^+})^{1^-} \rangle$.

Clearly, $P^i (1 \leq i \leq q)$ is a path of order $l = 4 \cdot \frac{l}{4}$, which contains $\frac{l}{2}$ neighbors of u. Then $F = \{P^1, P^2, \cdots, P^q\}$ forms a P_l-structure-cut of Q_n^k and $|F| = \frac{4n}{l}$.

Case 1.2. $2 \leq r \leq \frac{l-4}{2}$. We set

$P^1 = \langle (u)^{1^-}, ((u)^{1^-})^{0^+}, (u)^{0^+}, ((u)^{0^+})^{2^-}, \cdots, (u)^{(\frac{l}{4})^-}, ((u)^{(\frac{l}{4})^-})^{(\frac{l}{4}-1)^+}, (u)^{(\frac{l}{4}-1)^+}, ((u)^{(\frac{l}{4}-1)^+})^{(\frac{l}{4}+1)^-} \rangle$,

$P^2 = \langle (u)^{(\frac{l}{4}+1)^-}, ((u)^{(\frac{l}{4}+1)^-})^{(\frac{l}{4})^+}, (u)^{(\frac{l}{4})^+}, \cdots, (u)^{(\frac{l}{2})^-}, ((u)^{(\frac{l}{2})^-})^{(\frac{l}{2}-1)^+}, (u)^{(\frac{l}{2}-1)^+}, ((u)^{(\frac{l}{2}-1)^+})^{(\frac{l}{2}+1)^-} \rangle$,

...

$P^q = \langle (u)^{(n-\frac{r}{2}-\frac{l}{4}+1)^-}, ((u)^{(n-\frac{r}{2}-\frac{l}{4}+1)^-})^{(n-\frac{r}{2}-\frac{l}{4})^+}, (u)^{(n-\frac{r}{2}-\frac{l}{4})^+}, \cdots, (u)^{(n-\frac{r}{2})^-}, ((u)^{(n-\frac{r}{2})^-})^{(n-\frac{r}{2}-1)^+}, (u)^{(n-\frac{r}{2}-1)^+}, ((u)^{(n-\frac{r}{2}-1)^+})^{(n-\frac{r}{2}+1)^-} \rangle$,

$P^{(q+1)} = \langle (u)^{(n-\frac{r}{2}+1)^-}, ((u)^{(n-\frac{r}{2}+1)^-})^{(n-\frac{r}{2})^+}, (u)^{(n-\frac{r}{2})^+}, \cdots, (u)^{0^-}, ((u)^{0^-})^{(n-1)^+}, (u)^{(n-1)^+}, ((u)^{(n-1)^+})^{1^-}, (((u)^{(n-1)^+})^{1^-})^{2^+}, \cdots, ((((u)^{(n-1)^+})^{1^-})^{2^+\cdots})^{(l+1-2r(modn))^-} \rangle$.

Then $F = \{P^1, P^2, \cdots, P^q, P^{(q+1)}\}$ forms a P_l-structure-cut of Q_n^k and $|F| = \lceil \frac{4n}{l} \rceil$. When $l \equiv 2(mod4)$, one can similarly obtain the result.

Thus, $\kappa(Q_n^k; P_l) \leq \lceil \frac{4n}{l} \rceil$ and consequently $\kappa^s(Q_n^k; P_l) \leq \lceil \frac{4n}{l} \rceil$ when l is even. □

Lemma 3.9. *Let $3 \leq l \leq 2n, k \geq 4$ and $n \geq 5$. Then $\kappa(Q_n^k; P_l) \geq \lceil \frac{4n}{l+1} \rceil$ and $\kappa^s(Q_n^k; P_l) \geq \lceil \frac{4n}{l+1} \rceil$ if l is odd; $\kappa(Q_n^k; P_l) \geq \lceil \frac{4n}{l} \rceil$ and $\kappa^s(Q_n^k; P_l) \geq \lceil \frac{4n}{l} \rceil$ if l is even.*

Proof. If no confusion should arise, we use $F = \{\underbrace{P_1, \cdots, P_1}_{\lambda_1}, \underbrace{P_2, \cdots, P_2}_{\lambda_2}, \cdots, \underbrace{P_l, \cdots, P_l}_{\lambda_l}\}$ to denote a set of connected subgraphs of P_l with $|F| = \sum_{i=1}^{l} \lambda_i$ for $\lambda_i \geq 0$. We prove this by contradiction.

Case 1. When l is odd, let $|F| \leq \lceil \frac{4n}{l+1} \rceil - 1$. Suppose to the contrary that $Q_n^k - F$ is disconnected, then $Q_n^k - F$ has at least two components. Without loss of generality, let C be a smallest component of $Q_n^k - F$. We consider the following three events.

Case 1.1. $|V(C)| = 1$. We set $V(C) = \{u\}$, then $|N_{Q_n^k}(u)| = 2n$. By Lemma 3.3, every element in F contains at most $\frac{l+1}{2}$ neighbours of u. Thus, we have to delete at least $\lceil \frac{2n}{\frac{l+1}{2}} \rceil = \lceil \frac{4n}{l+1} \rceil$ elements of F to isolate C. But it is impossible since $|F| \leq \lceil \frac{4n}{l+1} \rceil - 1 < \lceil \frac{4n}{l+1} \rceil$.

Case 1.2. $|V(C)| \geq 2$. By Lemma 3.6, $\kappa_1(Q_n^k) = 4n - 2$. This implies that we have to delete at least $4n - 2$ nodes to isolate C. However, from the assumption $|F| \leq \lceil \frac{4n}{l+1} \rceil - 1$, we infer that $|V(F)| \leq l(\lceil \frac{4n}{l+1} \rceil - 1) \leq l(\frac{4n+l-1}{l+1} - 1) = \frac{l}{l+1}(4n-2) < 4n-2$, a contradiction.

Case 2. l is even. Let $|F| \leq \lceil \frac{4n}{l} \rceil - 1$. With similar argument as that in the proof of Case 1, we consider the following three events.

Case 2.1. $|V(C)| = 1$. We set $V(C) = \{u\}$, then $|N_{Q_n^k}(u)| = 2n$. By Lemma 3.3, every element in F contains at most $\frac{l}{2}$ neighbours of u. Thus, we have to delete at least $\lceil \frac{2n}{\frac{l}{2}} \rceil = \lceil \frac{4n}{l} \rceil$ elements of F to isolate C. But it is impossible since $|F| \leq \lceil \frac{4n}{l} \rceil - 1 < \lceil \frac{4n}{l} \rceil$.

Case 2.2. $|V(C)| = 2$. Suppose that $V(C) = \{\{u, v\} | (u, v) \in E(Q_n^k)\}$, then $|N_{Q_n^k}(u, v)| = 4n - 2$. By Lemma 3.5, every element in F contains at most $l - 1$ neighbours of $\{u, v\}$. It means that we have to delete at least $\lceil \frac{4n-2}{l-1} \rceil$ elements of F to isolate C. Then, $|F| \leq \lceil \frac{4n}{l} \rceil - 1 \leq \frac{4n+l-2}{l} - 1 = \frac{4n-2}{l} < \frac{4n-2}{l-1} \leq \lceil \frac{4n-2}{l-1} \rceil$, a contradiction.

Case 2.3. $|V(C)| \geq 3$. By Lemma 3.7, $\kappa_2(Q_n^k) = 6n - 5$ when $k \geq 4$ and $n \geq 5$. Thus, we have to delete at least $6n - 5$ nodes to isolate C. However, by the assumption $|F| \leq \lceil \frac{4n}{l} \rceil - 1$. We have $|V(F)| \leq l(\lceil \frac{4n}{l} \rceil - 1) \leq l(\frac{4n+l-2}{l} - 1) = 4n - 2 < 6n - 5$ for $n \geq 5$, a contradiction. □

Combining Lemma 3.8 and Lemma 3.9, we have the following theorem.

Theorem 3.10. *Let $3 \leq l \leq 2n, k \geq 4$ and $n \geq 5$. Then $\kappa(Q_n^k; P_l) = \lceil \frac{4n}{l+1} \rceil$ and $\kappa^s(Q_n^k; P_l) = \lceil \frac{4n}{l+1} \rceil$ if l is odd; $\kappa(Q_n^k; P_l) = \lceil \frac{4n}{l} \rceil$ and $\kappa^s(Q_n^k; P_l) = \lceil \frac{4n}{l} \rceil$ if l is even.*

Please cite this article in press as: L. Miao et al., Structure fault tolerance of k-ary n-cube networks, Theoret. Comput. Sci. (2019), https://doi.org/10.1016/j.tcs.2019.06.013

4. $\kappa(Q_n^k; C_l)$ and $\kappa^s(Q_n^k; C_l)$

Note that the k-ary n-cube Q_n^k is bipartite if k is even, thus it contains no odd cycles. In the following, we consider $\kappa(Q_n^k; C_l)$ and $\kappa^s(Q_n^k; C_l)$ when l is even.

Lemma 4.1. *Let $6 \le l \le 2n$. Then $\kappa(Q_n^k; C_l) \le \lceil \frac{4n}{l} \rceil$ and $\kappa^s(Q_n^k; C_l) \le \lceil \frac{4n}{l} \rceil$ if l is even.*

Proof. Without loss of generality, we assume that $l \equiv 0(mod4)$. Let $2n = q \cdot (\frac{l}{2}) + r$ for some nonnegative integers q and r with $0 \le r \le \frac{l-2}{2}$. Since $2n \ge l$, we have $q \ge 1$. We set u being an arbitrary node in Q_n^k.

Case 1.1. $r = 0$. If $l = 8$. We set

$C^1 = \langle (u)^{1^-}, ((u)^{1^-})^{0^+}, (u)^{0^+}, ((u)^{0^+})^{2^-}, (u)^{2^-}, ((u)^{2^-})^{0^-}, (u)^{0^-}, ((u)^{0^-})^{1^-}, (u)^{1^-} \rangle$,

$C^2 = \langle (u)^{3^-}, ((u)^{3^-})^{2^+}, (u)^{2^+}$, MATH, $(u)^{4^-}, ((u)^{4^-})^{1^+}, (u)^{1^+}, ((u)^{1^+})^{3^-}, (u)^{3^-} \rangle$,

...

$C^{\frac{n}{2}} = \langle (u)^{(n-1)^-}, ((u)^{(n-1)^-})^{(n-2)^+}, (u)^{(n-2)^+}, ((u)^{(n-2)^+})^{(n-1)^+}, (u)^{(n-1)^+}, ((u)^{(n-1)^+})^{(n-3)^+}, (u)^{(n-3)^+}, ((u)^{(n-3)^+})^{(n-1)^-}, (u)^{(n-1)^-} \rangle$.

Then, $C^i (1 \le i \le \frac{n}{2})$ is a cycle of order $l = 8$, which contains 4 neighbors of u. Then $F = \{C^1, C^2, \cdots, C^{\frac{n}{2}}\}$ forms a C_l-structure-cut of Q_n^k and $|F| = \frac{4n}{l} = \frac{n}{2}$.

Otherwise, we set

$C^1 = \langle (u)^{1^-}, ((u)^{1^-})^{0^+}, (u)^{0^+}, ((u)^{0^+})^{2^-}, \cdots, (u)^{(\frac{l}{4})^-}, ((u)^{(\frac{l}{4})^-})^{(\frac{l}{4}-1)^+}, (u)^{(\frac{l}{4}-1)^+}, ((u)^{(\frac{l}{4}-1)^+})^{1^-}, (u)^{1^-} \rangle$,

$C^2 = \langle (u)^{(\frac{l}{4}+1)^-}, ((u)^{(\frac{l}{4}+1)^-})^{(\frac{l}{4})^+}, (u)^{(\frac{l}{4})^+}, \cdots, (u)^{(\frac{l}{2})^-}, ((u)^{(\frac{l}{2})^-})^{(\frac{l}{2}-1)^+}, (u)^{(\frac{l}{2}-1)^+}, ((u)^{(\frac{l}{2}-1)^+})^{(\frac{l}{4}+1)^-}, (u)^{(\frac{l}{4}+1)^-} \rangle$,

...

$C^q = \langle (u)^{(n-\frac{l}{4}+1)^-}, ((u)^{(n-\frac{l}{4}+1)^-})^{(n-\frac{l}{4})^+}, \cdots, (u)^{0^-}, ((u)^{0^-})^{(n-1)^+}, (u)^{(n-1)^+}, ((u)^{(n-1)^+})^{(n-\frac{l}{4}+1)^-}, (u)^{(n-\frac{l}{4}+1)^-} \rangle$.

Obviously, $C^i (1 \le i \le q)$ is a cycle of order $l = 4 \cdot \frac{l}{4}$, which contains $\frac{l}{2}$ neighbors of u. Then $F = \{C^1, C^2, \cdots, C^q\}$ forms a C_l-structure-cut of Q_n^k and $|F| = \frac{4n}{l}$.

Case 1.2. $2 \le r \le \frac{l-4}{2}$. If $l = 8$, then $r = 2$ and we set

$C^1 = \langle (u)^{1^-}, ((u)^{1^-})^{0^+}, (u)^{0^+}, ((u)^{0^+})^{2^-}, (u)^{2^-}, ((u)^{2^-})^{0^-}, (u)^{0^-}, ((u)^{0^-})^{1^-}, (u)^{1^-} \rangle$,

$C^2 = \langle (u)^{3^-}, ((u)^{3^-})^{2^+}, (u)^{2^+}$, MATH, $(u)^{4^-}, ((u)^{4^-})^{1^+}, (u)^{1^+}, ((u)^{1^+})^{3^-}, (u)^{3^-} \rangle$,

...

$C^{\frac{n-1}{2}} = \langle (u)^{(n-2)^-}, ((u)^{(n-2)^-})^{(n-3)^+}, (u)^{(n-3)^+}, ((u)^{(n-3)^+})^{(n-1)^-}, (u)^{(n-1)^-}, ((u)^{(n-1)^-})^{(n-4)^+}, (u)^{(n-4)^+}, ((u)^{(n-4)^+})^{(n-2)^-}, (u)^{(n-2)^-} \rangle$,

$C^{\frac{n+1}{2}} = \langle (u)^{(n-2)^+}, ((u)^{(n-2)^+})^{(n-1)^+}, (u)^{(n-1)^+}, ((u)^{(n-1)^+})^{(n-3)^-}, (((u)^{(n-1)^+})^{(n-3)^-})^{(n-2)^+}, ((u)^{(n-3)^-})^{(n-2)^+}, (((u)^{(n-3)^-})^{(n-2)^+})^{(n-1)^-}, ((u)^{(n-2)^+})^{(n-1)^-}, (u)^{(n-2)^+} \rangle$.

Now $F = \{C^1, C^2, \cdots, C^{\frac{n+1}{2}}\}$ forms a C_l-structure-cut of Q_n^k and $|F| = \lceil \frac{n}{2} \rceil = \lceil \frac{4n}{l} \rceil$.

Otherwise, we set

$C^1 = \langle (u)^{1^-}, ((u)^{1^-})^{0^+}, (u)^{0^+}, ((u)^{0^+})^{2^-}, \cdots, (u)^{(\frac{l}{4})^-}, ((u)^{(\frac{l}{4})^-})^{(\frac{l}{4}-1)^+}, (u)^{(\frac{l}{4}-1)^+}, ((u)^{(\frac{l}{4}-1)^+})^{1^-}, (u)^{1^-} \rangle$,

$C^2 = \langle (u)^{(\frac{l}{4}+1)^-}, ((u)^{(\frac{l}{4}+1)^-})^{(\frac{l}{4})^+}, (u)^{(\frac{l}{4})^+}, \cdots, (u)^{(\frac{l}{2})^-}, ((u)^{(\frac{l}{2})^-})^{(\frac{l}{2}-1)^+}, (u)^{(\frac{l}{2}-1)^+}, ((u)^{(\frac{l}{2}-1)^+})^{(\frac{l}{4}+1)^-}, (u)^{(\frac{l}{4}+1)^-} \rangle$,

...

$C^q = \langle (u)^{(n-\frac{r}{2}-\frac{l}{4}+1)^-}, ((u)^{(n-\frac{r}{2}-\frac{l}{4}+1)^-})^{(n-\frac{r}{2}-\frac{l}{4})^+}, (u)^{(n-\frac{r}{2}-\frac{l}{4})^+}, \cdots, (u)^{(n-\frac{r}{2})^-}, ((u)^{(n-\frac{r}{2})^-})^{(n-\frac{r}{2}-1)^+}, (u)^{(n-\frac{r}{2}-1)^+}, ((u)^{(n-\frac{r}{2}-1)^+})^{(n-\frac{r}{2}-\frac{l}{4}+1)^-}, (u)^{(n-\frac{r}{2}-\frac{l}{4}+1)^-} \rangle$,

$C^{(q+1)} = \langle (u)^{(n-\frac{r}{2}+1)^-}, ((u)^{(n-\frac{r}{2}+1)^-})^{(n-\frac{r}{2})^+}, (u)^{(n-\frac{r}{2})^+}, \cdots, (u)^{0^-}, ((u)^{0^-})^{(n-1)^+}, (u)^{(n-1)^+}, P, (((((u)^{(n-1)^+})^{1^-})^{2^+ \cdots})^{(\frac{l}{2}-r)^+})^{(n-\frac{r}{2}+1)^-}, Q^{-1} \rangle$,

where $P = \langle (((u)^{(n-1)^+})^{1^-}, (((u)^{(n-1)^+})^{1^-})^{2^+}, \cdots, ((((u)^{(n-1)^+})^{1^-})^{2^+ \cdots})^{(\frac{l}{2}-r)^+} \rangle$, $Q = \langle (u)^{(n-\frac{r}{2}+1)^-}, ((u)^{(n-\frac{r}{2}+1)^-})^{1^-}, \cdots, ((((u)^{(n-\frac{r}{2}+1)^-})^{1^-})^{2^+ \cdots})^{(\frac{l}{2}-r)^+} \rangle$.

Then $F = \{C^1, C^2, \cdots, C^q, C^{(q+1)}\}$ forms a C_l-structure-cut of Q_n^k and $|F| = \lceil \frac{4n}{l} \rceil$.

Thus, $\kappa(Q_n^k; C_l) \le \lceil \frac{4n}{l} \rceil$ and consequently $\kappa^s(Q_n^k; C_l) \le \lceil \frac{4n}{l} \rceil$ when l is even. □

Lemma 4.2. *Let $6 \le l \le 2n$, $k \ge 4$ and $n \ge 5$. Then $\kappa(Q_n^k; C_l) \ge \lceil \frac{4n}{l} \rceil$ and $\kappa^s(Q_n^k; C_l) \ge \lceil \frac{4n}{l} \rceil$ if l is even.*

Proof. Let $F = \{\underbrace{P_1, \cdots, P_1}_{\lambda_1}, \underbrace{P_2, \cdots, P_2}_{\lambda_2}, \cdots, \underbrace{P_l, \cdots, P_l}_{\lambda_l}, \underbrace{C_l, \cdots, C_l}_{\mu}\}$ and $|F| = \sum_{i=1}^{l} \lambda_i + \mu \le \lceil \frac{4n}{l} \rceil - 1$ for $\lambda_i \ge 0, \mu \ge 0$. Suppose to the contrary that $Q_n^k - F$ is disconnected, then $Q_n^k - F$ has at least two components. Without loss of generality, let C be a smallest component of $Q_n^k - F$. The result can be proved by considering the similar three events as Lemma 3.9. □

By Lemma 4.1 and Lemma 4.2, we have the following theorem.

Theorem 4.3. *Let $6 \le l \le 2n$, $k \ge 4$ and $n \ge 5$. Then $\kappa(Q_n^k; C_l) = \lceil \frac{4n}{l} \rceil$ and $\kappa^s(Q_n^k; C_l) = \lceil \frac{4n}{l} \rceil$ if l is even.*

Please cite this article in press as: L. Miao et al., Structure fault tolerance of k-ary n-cube networks, Theoret. Comput. Sci. (2019), https://doi.org/10.1016/j.tcs.2019.06.013

Note that it's an open problem to analyze the (sub)structure connectivity of the k-ary n-cube for $k = 3$. One could attempt to modify the techniques used here to complete it.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The research is supported by NSFC (No. 11671296), SRF for ROCS, SEM and Fund Program for the Scientific Activities of Selected Returned Overseas Professionals in Shanxi Province.

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